TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 219, 1976

BREADTH TWO TOPOLOGICAL LATTICES WITH CONNECTED SETS OF IRREDUCIBLES

BY

J. W. LEA, JR.

ABSTRACT. Breadth two topological lattices with connected sets of irreducible elements are characterized by these sets.

1. **Introduction.** In the study of topological lattices and semilattices on the two cell, the boundary arcs have played an important role. A. D. Wallace [18], D. R. Brown [4], and A. Y. W. Lau [8] have given conditions under which the boundary generates the lattice or semilattice. J. B. Rhodes [16] has shown that two such lattices are iseomorphic (homeomorphic and isomorphic) if their boundaries are order isomorphic. Rhodes, for metric lattices, and the author, in the general case, have found that the meet and join irreducible elements are contained in the boundary [16], [13].

In this article we establish conditions for the boundary of a topological lattice on the two cell to consist entirely of irreducible elements. These lattices may then be characterized by their chains of irreducible elements.

In §4 we shall extend the study to nondistributive lattices of breadth two.

The only use made of the metric condition is that any two nondegenerate arc chains (compact connected chains) are iseomorphic. Thus the results are valid in any class of lattices having this property.

2. **Preliminaries.** The *width* $w(X)$ of a partially ordered set X is the maximum number of elements in a set of incomparable elements. For $x \in X$ we let $M(x) = \{y \in X: x \leqslant y\}$ and $L(x) = \{y \in X: y \leqslant x\}$; when $x \leqslant y$, then $[x, y] = M(x) \cap L(y)$.

Let L be a bounded lattice and C a chain in L which contains 1 and which is closed under arbitrary meets in L. Then $s_C: L \longrightarrow C$ is defined by $s_C(x) = \bigwedge\{y \in C: x \leqslant y\}$ for all $x \in L$. In [2] it is shown that s_C is a join homomorphism, $x \leqslant s_C(x)$ for all $x \in L$, $s_C(p) = p$ for all $p \in C$; also s_C is a meet homomorphism when C consists of prime elements. We shall use t_C for the dual notion.

Received by the editors October 23, 1974 and, in revised form, May 28, 1975.

AMS (MOS) subject classifications (1970). Primary 06A30, 06A35, 54F05; Secondary 22A30.

Key words and phrases. Topological lattice, meet (join) irreducible.

We let $M(L)$ $[J(L)]$ denote the set of all meet [join] irreducible elements of L; also A^* and A° denote, respectively, the topological closure and interior of A.

3. **Distributive lattices.** The two goals of this section are to establish conditions under which $M(L) \cup J(L)$ is the whole boundary of L and to characterize L by $M(L) \cup J(L)$.

THEOREM 3.1. *Let L be a compact topological lattice of finite breadth with $M(L)$ connected. If no meet irreducible is a cutpoint of L, then L has no cutpoints.*

PROOF. If the breadth of L is n, then the continuous function $(M(L))^n \longrightarrow L$ defined by $(x_1, \ldots, x_n) \mapsto x_1 \wedge \cdots \wedge x_n$ is onto [2]. Thus L is connected. Suppose $p \in L$ is a cutpoint. Then $L = L(p) \cup M(P)$ [1]. Since $1 \in M(L)$, then $M(L) \cap (M(p)\backslash\{p\}) \neq \emptyset$. Let $x \in L(p)\backslash\{p\}$ and let $x = x_1 \wedge \cdots \wedge x_n$ with all $x_i \in M(L)$. For some i, we have $x_i \in L(p)\backslash\{p\}$; hence $M(L) \cap (L(p)\backslash\{p\}) \neq \emptyset$. Thus $M(L)$ has a nontrivial separation. But $M(L)$ is connected; therefore L has no cutpoints.

THEOREM 3.2. *Let L be a compact connected topological lattice of finite breadth with $w(M(L)) = 2$. Then L is iseomorphic to a sublattice of a direct product of two arc chains.*

PROOF. The breadth of L is two [12] and so L is modular [6]. Thus L is distributive [3, p. 66]; hence the conclusion follows [2].

A lattice has meet (join) representations if each element is a meet (join) of meet (join) irreducibles.

THEOREM 3.3. *Let L be a bounded distributive lattice of breadth two which has finite irredundant meet and join representations and for which $w(M(L)) = 2$. Then $M(L) \cap J(L) \neq \emptyset$.*

PROOF. If $0 \in M(L)$, we are done. Suppose $\{x, y\} \subset M(L)$ is meet irredundant and $0 = x \wedge y$. Suppose also that $x = a \vee b$ with $a, b \in J(L)$. If $a = a_1 \wedge a_2$ and $b = b_1 \wedge b_2$ with a_1, a_2, b_1, and $b_2 \in M(L)$, then $0 = x \wedge y \geqslant a \wedge y = a_1 \wedge a_2 \wedge y$; thus $0 = a_1 \wedge y$ or $0 = a_2 \wedge y$ or $0 = a_1 \wedge a_2$. If the latter case holds we are done since then $x = b \in J(L)$. Thus suppose $0 = a_1 \wedge y$. Since L is distributive, $x = a_1$ [3, p. 58]. In a similar manner $x = b_1$ or b_2; say $x = b_2$. Since $w(M(L)) = 2$, two of a_1, a_2, b_1 must be comparable. It is now easily verified that a, b, or x must belong to $M(L) \cap J(L)$.

DEFINITION 3.4. If L is a lattice with 1 and C is a chain in L, then C is a *coordinate chain* of L if

(i) $1 \in C$,

(ii) C is closed under arbitrary meets in L, and

(iii) $x \in C$ implies $x = \bigwedge D$ for some $D \subset M(L) \cap C$.

A *marginal* element of a space is an element of the cohomological boundary (see [10]).

THEOREM 3.5. *Let L be a compact connected distributive topological lattice of breadth two. Let A and B be coordinate chains for L and C and D their respective extensions to maximal chains from* 0 *to* 1. *If $p \notin C \cup D$, then p is not marginal in L.*

PROOF. Let $p \in L\backslash(C \cup D)$. Then $p = s_A(p) \wedge s_B(p)$. Under the iseomorphism mentioned in Theorem 3.2, the image of p in $A \times B$ is $(s_A(p), s_B(p)) \in (\mathrm{Im}(L))^\circ$. Since $p \notin A \cup B$, then $s_A(p)$ $(s_B(p))$ is not marginal in A (B). Therefore $(s_A(p), s_B(p))$ is not marginal in $A \times B$ [10]. Thus p is not marginal in L.

K. Oberhoff has obtained a result similar to Theorem 3.5 assuming that the underlying space of L is the product of two arc chains [15].

THEOREM 3.6. *Let L be a compact topological lattice of finite breadth with no meet irreducible cutpoints. If $M(L)$ and $J(L)$ are both the union of two arc chains, then the set B of all marginal elements of L is equal to $M(L) \cup J(L)$.*

PROOF. That $M(L) \cup J(L) \subset B$ follows from Theorem 3.1 above and Theorem 2.1 of [13]. Let C and D be coordinate chains and extend C and D to maximal chains S and T from 0 to 1. Let $x \in B\backslash\{0, 1\}$. By Theorem 3.5, $x \in S \cup T$. Suppose $x \in S\backslash(M(L) \cup J(L))$. By Theorem 3.3, there exists an element $y \in M(L) \cap J(L) \cap S$. Either $x < y$ or $y < x$. If $x < y$, let $x = s \vee t$ with $s \in J(L) \cap S$ and $t \in J(L) \cap T$. Then $[J(L) \cap S \cap L(X)] \cup [J(L) \cap S \cap M(x)]$ is a nontrivial separation of the connected set $J(L) \cap S$. Thus $B \subset M(L) \cup J(L)$.

If S and T are the chains in Theorem 3.6, then it is easily verified that $L = S \wedge T = S \vee T$ and $S \cap T = \{0, 1\}$.

We shall now characterize the lattices of Theorem 3.6. If $A = M \cup J$ is an arc chain from 0 to 1 in a lattice L with M and J nondegenerate arc chains containing, respectively, 1 and 0, then we have exactly five possibilities:

P1. $A = M = J$;

P2. $A = M$ and $\bigvee J = p$ where $0 < p < 1$;

P3. $A = J$ and $\bigwedge M = p$ where $0 < \mathrm{p} < 1$;

P4. $M \cap J$ is an arc chain from p to q with $0 < p < q < 1$;

P5. $M \cap J = \{p\}$ and $0 < p < 1$.

DEFINITION 3.7. Let L be a compact distributive topological lattice of breadth two. Let A_1 and A_2 be arc chains from 0 to 1 satisfying for $i = 1, 2$:

(1) $A_i = M_i \cup J_i$;

(2) $M_i(J_i)$ is a nondegenerate arc chain containing 1 (0) and $M(L) = M_1 \cup M_2$ $[J(L) = J_1 \cup J_2]$;

(3) $A_1 \cap A_2 = \{0, 1\}$.

Then L will be called an *IC-lattice.*

For the remainder of this paper all lattices will be assumed metrizable.

THEOREM 3.8. *If L is an IC-lattice and both A_1 and A_2 satisfy* P5, *then L is iseomorphic to the direct product of two arc chains.*

PROOF. Clearly $p_1 \in M_1 \cap J_1$ and $p_2 \in M_2 \cap J_2$ are complements. The theorem is thus a direct application of Lemma 2 of [5].

We shall need the following lemma.

LEMMA 3.9. *Let L be a bounded topological lattice with A and B arc chains from* 0 *to* 1. *If*

(i) $s_B|A(x) = 0$ *if and only if* $x = 0$, *and*

(ii) $y \in M(x)^\circ$ *whenever* $x, y \in A$ *and* $x < y$,

then $s_B|A$ is one-to-one.

PROOF. If $s_B|A$ is not one-to-one, then for some $x, y \in A$ with $0 < x < y$ we have $s_B|A(x) = s_B|A(y)$. It follows easily that $(B \cap M(x)^\circ) \cup (B\backslash M(x))$ is a nontrivial separation of the connected set B.

An IC-lattice in which A_1 and A_2 both satisfy P1 is usually called a banana. An example in the plane is $L = \{(x, y)\colon 0 \leq x \leq 1, x^2 \leq y \leq \sqrt{x}\}$ with the usual topology and order of the plane.

When a chain has a subscript C_i, then we use s_i for s_{C_i}.

THEOREM 3.10. *If L and L' are IC-lattices and A_1, A_2, A_1', and A_2' all satisfy* P1, *then L is iseomorphic to L'.*

PROOF. The theorem is proved by explicitly constructing an order isomorphism of $A_1 \cup A_2$ onto $A_1' \cup A_2'$. It will then follow from Corollary 3.9 of [16] that L is iseomorphic to L'.

We begin the construction by partitioning the arcs A_i as follows. Let $q_0 \in A_2\backslash\{0, 1\}$. We define

$p_0 = t_1(q_0)$, and for $n = 1, 2, \ldots,$

$p_{2n-1} = t_1(t_2t_1)^n(q_0),$

$p_{2n} = s_1(s_2s_1)^{n-1}(q_0),$

$q_{2n-1} = (t_2t_1)^n(q_0)$, and

$q_{2n} = (s_2s_1)^n(q_0).$

With these points we partition A_1 into the following subarcs:

$P(01) = A_1 \cap [p_1, p_0]$,

$P(02) = A_1 \cap [p_0, p_2]$, and for $n = 1, 2, \ldots,$

$P(2n-1, 2n+1) = A_1 \cap [p_{2n+1}, p_{2n-1}]$, and
$P(2n, 2n+2) = A_1 \cap [p_{2n}, p_{2n+2}]$.

We define the partition of A_2 similarly using q_0, q_1, etc., to obtain $Q(01)$, $Q(02)$, etc. Primes are used to denote the corresponding sets and points of L'.

Let g: $P(02) \longrightarrow P(02)'$ be any iseomorphism onto $P(02)'$. We define f: $L \longrightarrow L'$ by

(i) $f | P(02) = g$;
(ii) $f(0) = 0, f(1) = 1, f(q_0) = q_0'$;
(iii) if $x \in Q(01)$, then $f(x) = t_2' g s_1(x)$;
(iv) if $x \in Q(02)$, then $f(x) = s_2' g t_1(x)$;
(v) if $x \in P(01)$, then $f(x) = t_1' t_2' g s_1 s_2(x)$.

This process may be continued inductively for each of the remaining subarcs of A_1 and A_2. By applying Lemma 3.9 above and Lemma 2.3 of [12] we see that the restrictions of the s and t functions to the A_i are iseomorphisms. Thus f maps a subarc of A_i iseomorphically onto the corresponding subarc of A_i'. Finally it is straightforward, and tedious, to verify that f is an order isomorphism on $A_1 \cup A_2$.

For the nonmetric case, we define f on $L \backslash (A_1 \cup A_2)$ by $f(x) = f(x_1) \wedge f(x_2)$, where $x = x_1 \wedge x_2$ is the unique representation of x as a meet of meet irreducibles. Then f is easily shown to be an isomorphism on L. To see that f is continuous we observe that for each interval $[a, b] \subset L'$,

$$f^{-1}([a, b]) = f^{-1}(M(a) \cap L(b)) = M(f^{-1}(a)) \cap L(f^{-1}(b));$$

hence $f^{-1}([a, b])$ is closed. Since L and L' have the interval topology [9], f is continuous.

Since each of the arcs A_1 and A_2 of an IC-lattice may satisfy any one of the conditions P1 to P5, there are twenty-five lattices possible. Eighteen of these possibilities yield lattices unique in the sense of Theorem 3.10. The seven exceptions are:

(1) A_1 satisfies P1 and A_2 satisfies P4;
(2) A_1 satisfies P2 and A_2 satisfies P4;
(3) A_1 satisfies P2 and A_2 satisfies P3;
(4) A_1 and A_2 both satisfy P4.

The other three cases are obtained by interchanging A_1 and A_2 in (1), (2), and (3). Eliminating those cases obtained by such interchanges leaves us with fifteen distinct lattices.

Suppose L is an IC-lattice with A_1 and A_2 as in case 1. Then $s_1(p)$ and $t_1(q)$ may be the same point or distinct points of L; see Figure 1. Thus the lattices in Figure 1 are not iseomorphic.

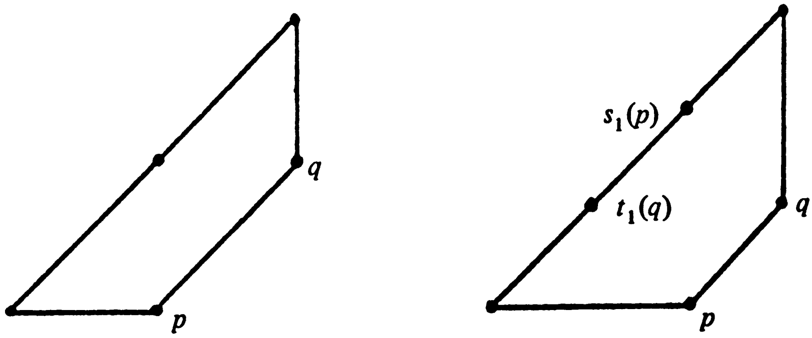

FIGURE 1

4. **Nondistributive lattices.** In this section we shall extend the methods of §3 to compact connected topological lattices of breadth two for which $w(M(L)) = 3$. Such lattices are modular [6] but nondistributive [2, 3.1], [12, 2.10]. Examples of such lattices have been constructed in the three cell by D. E. Edmondson [7].

The difficulty in extending the methods of §3 is the lack of uniqueness in the representation of elements as meets of two meet irreducibles. When $w(M(L)) = 3$, the set of neutral elements of L is an arc chain which can be used to recover the needed unique meet representations. An example will be given to show that if $w(M(L)) > 3$, then the arcs of irreducibles are not sufficient to characterize L.

DEFINITION 4.1. A lattice L is an *E-lattice* if L is a compact connected topological lattice of breadth two with $M(L) = \bigcup_{i=1}^{3} M_i$ where, for $i, j = 1, 2, 3$,

(i) M_i is an arc chain,

(ii) $J(L) = \bigcup_{i=1}^{3} J_i$ and J_i is an arc chain,

(iii) $A_i = M_i \cup J_i$ is an arc chain from 0 to 1,

(iv) $A_i \cap A_j = \{0, 1\}$ when $i \neq j$,

(v) $M_i \cap J_i = z_i \neq 0$,

(vi) $z_i \wedge z_j = 0$ when $i \neq j$.

In Definition 4.1 the chains A_i are the boundary arcs. Thus in an E-lattice all the A_i satisfy P5. As only minor modifications are needed to pass from this case to any other of the 125 possible cases, we shall present only the details of this case.

THEOREM 4.2. *If L is an E-lattice, then*

(i) $J_i = [0, z_i] \subset M_i \wedge M_j$ *when* $i \neq j$, *and*

(ii) $p \in L$ *implies* p *belongs to at least two of* $M_1 \wedge M_2$, $M_1 \wedge M_3$, *and* $M_2 \wedge M_3$.

PROOF. By Lemma 1.1 of [17], $[0, z_i]$ is a chain from 0 to z_i; hence $J_i = [0, z_i]$ [1]. Also $z_i \wedge M_j$ is an arc chain from 0 to z_i and so $J_i = z_i \wedge M_j \subset M_i \wedge M_j$ when $i \neq j$.

We note that for $i \neq j$, $M_i \wedge M_j$ is a distributive sublattice of L [14].

Suppose $p \in M_1 \wedge M_2$ and $p \notin (M_1 \wedge M_3) \cup (M_2 \wedge M_3)$. Letting $x_i = s_i(p)$ we have $p = x_1 \wedge x_2 \wedge x_3 = x_1 \wedge x_2$; also $p < x_1 \wedge x_3$ and $p < x_2 \wedge x_3$. If $x_3 = z_3$, then $p < x_1 \wedge z_3 \in [0, z_3] \subset M_1 \wedge M_3$, contrary to the assumption that $p \notin M_1 \wedge M_3$. If $x_3 \neq z_3$, then there exists an $x \in M_3$ such that $x < x_3$; then $x_1 \wedge x_2 \wedge x < p$. Since L has breadth two, $x_1 \wedge x_2 \wedge x = x_1 \wedge x$ or $x_2 \wedge x$. Thus, since M_3 is order dense, there exists a net $\{x_\alpha\}_{\alpha \in \Delta}$ converging to x_3 such that $x_\alpha < x_3$ and $x_1 \wedge x_2 \wedge x_\alpha = x_1 \wedge x_\alpha$ or $x_2 \wedge x_\alpha$.

Thus, for some cofinal subset $\Gamma \subset \Delta$, $x_1 \wedge x_2 \wedge x_\alpha = x_1 \wedge x_\alpha$ for all $\alpha \in \Gamma$ or $x_1 \wedge x_2 \wedge x_\alpha = x_2 \wedge x_\alpha$ for all $\alpha \in \Gamma$. If $x_1 \wedge x_2 \wedge x_\alpha = x_1 \wedge x_\alpha$ for all $\alpha \in \Gamma$, then the net $\{(x_1 \wedge x_\alpha, p)\}_{\alpha \in \Gamma}$ converges to $(x_1 \wedge x_3, p)$. Thus $x_1 \wedge x_3 \leqslant p < x_1 \wedge x_3$. We conclude that $p \in (M_1 \wedge M_3) \cup (M_2 \wedge M_3)$.

THEOREM 4.4. *If L is an E-lattice and if $A = \bigcap_{i \neq j}(M_i \wedge M_j)$, then* (i) *$A$ is an arc chain from* 0 *to* 1, *and* (ii) *$x, y \in A$ and $x < y$ imply $y \in M(x)^\circ$.*

PROOF. Clearly A is a compact chain. By Theorem 4.2, $L = (M_1 \wedge M_2) \cup (M_2 \wedge M_3)$. Since L is unicoherent, then $(M_1 \wedge M_2) \cap (M_2 \wedge M_3)$ is connected. Again from Theorem 4.2, $L = (M_1 \wedge M_3) \cup [(M_1 \wedge M_2) \cap (M_2 \wedge M_3)]$; thus A is connected.

If $x, y \in A$ and $x < y$, then $s_i(x) < s_i(y)$. By Theorem 3.8, $M_i \wedge M_j$ is iseomorphic to $M_i \times M_j$. The image of y, $(s_i(y), s_j(y))$, is in the interior of $M((s_i(x), s_j(x)))$. Since $L = \bigcup(M_i \wedge M_j)$ $(i \neq j)$, it follows that $y \in M(x)^\circ$.

The function $t_A | M_i$ is an iseomorphism with inverse $s_i | A$ for $i = 1, 2, 3$. For convenience we drop the subscript A. Note that for each $x \in L$, both x and $t(s_i(x))$ belong to the same chain $M_j \wedge (s_i(x))$ for some $j \neq i$. Thus x and $t(s_i(x))$ are related.

THEOREM 4.5. *Let L be an E-lattice. If $x = x_1 \wedge x_3 = x_2 \wedge x_3 \neq x_1 \wedge x_2$, where $x_i = s_i(x)$, then $x = t(x_1) \wedge x_i$ and $t(x_1) = t(x_2)$.*

PROOF. If $t(x_1) \leqslant x$, then $x \in [t(x_1), x_1] \subset M_1 \wedge M_2$ contrary to $x \neq x_1 \wedge x_2$. Hence $x < t(x_1)$, and so $x = x_1 \wedge x_3 \leqslant t(x_1) \wedge x_3 \leqslant x_1 \wedge x_3 = x$. Similarly $x = t(x_2) \wedge x_3$. Since s_1 is an isomorphism on $M_1 \wedge M_3$ and $s_i(x_j) = 1$, we have $s_1(t(x_1)) = s_1(t(x_1)) \wedge s_1(x_3) = s_1(t(x_1) \wedge x_3) = s_1(t(x_2) \wedge x_3) = s_1(t(x_2))$. Thus $t(x_1) = t(x_2)$.

THEOREM 4.6. *If L and L' are two E-lattices, then L is iseomorphic to L'.*

PROOF. Let A, M_i, J_i, t, and s_i be the sets and functions of the preceding discussion for L with primes denoting the corresponding sets and functions for L'. Let $g\colon A \longrightarrow A'$ be any iseomorphism onto A'. We extend g to $f\colon L \longrightarrow L'$ by

(i) $f|A = g$,

(ii) if $x \in M_i$, then $f(x) = s_i'gt(x)$, and

(iii) if $x \in L\backslash[(\bigcup_{i=1}^{3} M_i) \cup A]$, then $f(x) = gt(x_i) \wedge s_i'gt(x_j)$, where $x = t(x_i) \wedge x_j = x_i \wedge x_j = s_i(x) \wedge s_j(x)$.

From Theorem 4.5 we see that f is independent of the representation $t(x_i) \wedge x_j$ chosen for x. The proof that f is an iseomorphism on each $M_i \wedge M_j$ is similar to the proof of Theorem 3.10. If $x, y \in L$, then $x, y \in M_i \wedge M_j$ for some $i \neq j$. Since f is an isomorphism on $M_i \wedge M_j$, then $f(x \vee y) = f(x) \vee f(y)$ and $f(x \wedge y) = f(x) \wedge f(y)$.

EXAMPLE 4.7. Let T_1 and T_2 be two copies of the unit square I_2. We let L be the topological space obtained by identifying corresponding points of the region on and between the arcs $y = x^2$ and $y = \sqrt{x}$. Thus L consists of five regions which we identify as follows. Let $A_i = \{(x, y) \in T_i\colon 0 \leqslant x \leqslant 1, \sqrt{x} \leqslant y \leqslant 1\}$ and $B_i = \{(x, y) \in T_i\colon 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant x^2\}$. The fifth region is $T_1 \cap T_2 = \{(x, y)\colon 0 \leqslant x \leqslant 1, x^2 \leqslant y \leqslant \sqrt{x}\}$. In §3 we showed that A_i and B_i are each iseomorphic to $P = \{(x, y) \in I_2\colon 0 \leqslant x \leqslant 1, x \leqslant y \leqslant 1\}$ and $Q = \{(x, y) \in I_2\colon 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant x\}$. Thus we may consider $A_1 \cup A_2$ to be a copy of I_2 with $A_1 = P$ and $A_2 = Q$. Similarly for $B_1 \cup B_2$ with $B_1 = Q$ and $B_2 = P$.

We now describe a meet operation for L; joins are defined similarly. Let $a, b \in L$.

Case 1. If $a, b \in T_i$, then $a \wedge b$ is the meet of a and b in T_i.

Case 2. If $a \in A_1$ and $b \in A_2$, then a and b are mapped to $P \cup Q$ and the meet of their images is mapped back to $A_1 \cup A_2$ to provide a meet for a and b. The procedure for $a \in B_1$ and $b \in B_2$ is similar.

Case 3. If $a \in A_1$ and $b \in B_2$, then we let b' be that point of B_1 which corresponds to b; e.g. (½, ¼) may be considered a point of B_1 or B_2. We compute $c = a \wedge b'$ in T_1. If $c \notin B_1$, then c is the meet of a and b in L. If $c \in B_1$, then the corresponding point of B_2 is the meet of a and b in L. The case of $a \in A_2$ and $b \in B_1$ is handled similarly.

If $M_i = \{(x, 1) \in T_i\colon 0 \leqslant x \leqslant 1\}$ and $M_i' = \{(1, y) \in T_i\colon 0 \leqslant y \leqslant 1\}$, then the M_i and M_i' are the arc chains of meet irreducibles of L. Also $(M_1 \wedge M_2) \cap (M_1' \wedge M_2') = \{0, 1\}$.

In Edmondson's example constructed from four copies of the unit interval, $\bigcap_{i \neq j}(M_i \wedge M_j)$ is an arc chain from 0 to 1 [7]. Thus the nice characterization for lattices L with $w(M(L)) \leqslant 3$ does not generalize directly to lattices L with $w(M(L)) > 3$.

Portions of this paper appeared in [11].

REFERENCES

1. L. W. Anderson, *On the distributivity and simple connectivity of plane topological lattices*, Trans. Amer. Math. Soc. 91 (1959), 102–112. MR 21 #1365.

2. K. A. Baker and A. R. Stralka, *Compact, distributive lattices of finite breadth*, Pacific J. Math. 34 (1970), 311–320. MR 44 #129.

3. G. Birkhoff, *Lattice theory*, 3rd ed., Amer. Math. Soc. Colloq. Publ., vol. 25, Amer. Math. Soc., Providence, R. I., 1967. MR 37 #2638.

4. D. R. Brown, *Topological semilattices on the two-cell*, Pacific J. Math. 15 (1965), 35–46. MR 31 #725.

5. T. H. Choe, *Locally compact lattices with small lattices*, Michigan Math. J. 18 (1971), 81–85.

6. D. E. Edmondson, *Modularity in topological lattices*, Proc. Amer. Math. Soc. 21 (1969), 81–82. MR 39 #2132.

7. ———, *A modular topological lattice*, Pacific J. Math. 29 (1969), 271–277. MR 39 #4062.

8. A. Y. W. Lau, *Remarks on semilattices on the two-cell* (preprint).

9. J. D. Lawson, *Intrinsic topologies in topological lattices and semilattices*, Pacific J. Math. 44 (1973), 593–602. MR 47 #6580.

10. J. D. Lawson and B. Madison, *Peripheral and inner points*, Fund. Math. 69 (1970), 253–266. MR 43 #1175.

11. J. W. Lea, Jr., *Irreducible elements in compact topological lattices*, Dissertation, Louisiana State Univ., Baton Rouge, Louisiana, 1971.

12. ———, *An embedding theorem for compact semilattices*, Proc. Amer. Math. Soc. 34 (1972), 325–331.

13. ———, *The peripherality of irreducible elements of a lattice*, Pacific J. Math. 45 (1973), 555–560. MR 47 #4875.

14. ———, *Sublattices generated by chains in modular topological lattices*, Duke Math. J. 41 (1974), 241–246.

15. K. Oberhoff, *Semilattice structures on certain non-metric continua*, Dissertation, Univ. of Houston, Houston, Texas, 1973.

16. J. B. Rhodes, *Decomposition of semilattices with applications to topological lattices*, Pacific J. Math. 44 (1973), 299–307. MR 47 #3262.

17. E. D. Shirley and A. R. Stralka, *Homomorphisms on connected topological lattices*, Duke Math. J. 38 (1971), 483–490. MR 43 #5497.

18. A. D. Wallace, *Factoring a lattice*, Proc. Amer. Math. Soc. 9 (1958), 250–252. MR 20 #825.

DEPARTMENT OF MATHEMATICS, MIDDLE TENNESSEE STATE UNIVERSITY, MURFREESBORO, TENNESSEE 37132